Physics 402
Spring 2022
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Discussion Worksheet for October 10, 2022

1. The Slater determinant is a very handy way to construct antisymmetric wavefunctions of N-identical particle systems. Suppose you want to distribute particles into states *a*, *b*, *c*, etc. One forms rows of a determinant made up of $\psi_a(1)\quad \psi_b(1)\quad \psi_c(1)\ \ldots$ followed by the next row, written as $\psi_a(2)\quad \psi_b(2)\quad \psi_c(2)\ \ldots$, where "1" and "2" represent the coordinates of particle 1, particle 2, etc. Multiply the determinant by $1/\sqrt{N!}$ for normalization.

a) Form the antisymmetric wavefunction for two identical particles in states *a* and *b*.
b) Form the antisymmetric wavefunction for three identical particles in states *a, b* and *c*.
c) For the three identical particle case, see what happens if *a* and *c* are the same state.

a) $N=2$ $\quad \Psi_A(1,2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \psi_a(1) & \psi_b(1) \\ \psi_a(2) & \psi_b(2) \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right)$

b) $N=3$ $\quad \Psi_A(1,2,3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_a(1) & \psi_b(1) & \psi_c(1) \\ \psi_a(2) & \psi_b(2) & \psi_c(2) \\ \psi_a(3) & \psi_b(3) & \psi_c(3) \end{vmatrix}$

$$= \frac{1}{\sqrt{6}}\left\{ \psi_a(1)\begin{vmatrix} \psi_b(2) & \psi_c(2) \\ \psi_b(3) & \psi_c(3) \end{vmatrix} - \psi_b(1)\begin{vmatrix} \psi_a(2) & \psi_c(2) \\ \psi_a(3) & \psi_c(3) \end{vmatrix} + \psi_c(1)\begin{vmatrix} \psi_a(2) & \psi_b(2) \\ \psi_a(3) & \psi_b(3) \end{vmatrix} \right\}$$

$$\Psi_A(1,2,3) = \frac{1}{\sqrt{6}}\Big\{ \psi_a(1)\left[\psi_b(2)\psi_c(3) - \psi_b(3)\psi_c(2)\right] - \psi_b(1)\left[\psi_a(2)\psi_c(3) - \psi_a(3)\psi_c(2)\right] + \psi_c(1)\left[\psi_a(2)\psi_b(3) - \psi_a(3)\psi_b(2)\right]\Big\}$$

c) If states a and c are the same, call them a, what happens?

$$\Psi_A(1,2,3) \to \frac{1}{\sqrt{6}}\Big\{ \psi_a(1)\psi_b(2)\psi_a(3) - \psi_a(1)\psi_b(3)\psi_a(2) - \psi_b(1)\psi_a(2)\psi_a(3) + \psi_b(1)\psi_a(3)\psi_a(2) + \psi_a(1)\psi_a(2)\psi_b(3) - \psi_a(1)\psi_a(3)\psi_b(2)\Big\}$$

$= 0$

All 6 terms cancel pairwise.
There cannot be double occupation of any state!

2. Consider a spin-1/2 particle. It is known to be in the "up" state after a measurement of S_z. Show that in this state $\langle S_x \rangle = \langle S_y \rangle = 0$. Explain this result geometrically.

The particle is in the $S_z \to +\frac{\hbar}{2}$ eigenstate $\begin{pmatrix}1\\0\end{pmatrix} = \chi$

$$\langle S_x \rangle = \chi^+ S_x \chi = \begin{pmatrix}1 & 0\end{pmatrix} \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0$$

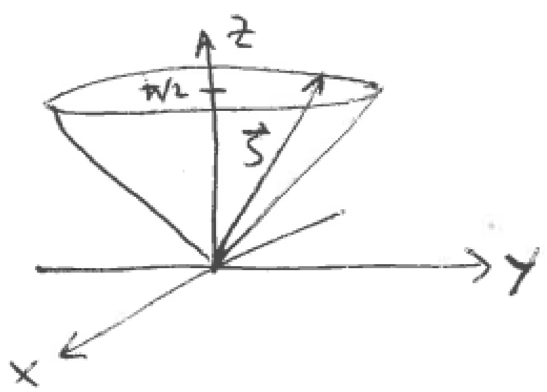
$$\langle S_y \rangle = \chi^+ S_y \chi = \begin{pmatrix}1 & 0\end{pmatrix} \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix} = 0$$

We know that $s = 1/2$. The S^2 eigenvalue is $\frac{1}{2}(1+\frac{1}{2})\hbar^2 = \frac{3}{4}\hbar^2$

Hence $\vec{S}$ has a length of $\frac{\sqrt{3}}{2}\hbar$

We know the z-component of $\vec{S}$ is $\hbar/2$.

The $\vec{S}$ vector lies somewhere (everywhere?) on a cone of height $\hbar/2$, inscribed in a sphere of radius $\frac{\sqrt{3}}{2}\hbar$



One interpretation is that the $\vec{S}$ vector is equally likely to be at any position on this cone, so an expectation value of the projection into the x-y plane will be zero.